Note: $B_r(A) = \{x \in X : d(x,a) < r\}$. $C[0,1]$ is the set of continous real valued functions on $[0,1]$ with sup metric.

Q1. *(15 marks)* Determine the nature of the critical points of $f(x,y) = x^3+6xy+3y^2-9x$.

Solution: Consider the partial derivatives:

$$\frac{\partial f}{\partial x} = 3x^2 + 6y - 9$$
$$\frac{\partial f}{\partial y} = 6x + 6y$$
$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x,\ f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6,\ \text{and } f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 6.$$

The critical points are given by

$$\frac{\partial f}{\partial x} = 3x^2 + 6y - 9 = 0$$
$$\frac{\partial f}{\partial y} = 6x + 6y = 0$$
$$\implies \quad \text{The critical points are } (-1,1) \text{ and } (3,-3).$$

The Hessian is given by $f_{xx}f_{yy} - f_{xy}^2 = 36(x-1)$.

At $(3,-3)$, the Hessian and f_{xx} are positive and hence the function attains local minima.
At $(-1,1)$, the Hessian is negative and hence the function attains local maxima.

Q2. *(10+5 marks)* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Compute the partial derivatives of f at $(0,0)$.

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{\frac{h^2.0}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0} \frac{\frac{0.h}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

(b) Prove that f is not differentiable at $(0,0)$.

Solution: Suppose the function is differentiable at the origin then the partial derivatives must exist and be continuous at the origin.
Note that the partial derivatives at $(x, y) \neq (0, 0)$ are given by

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{2xy^3}{(x^2 + y^2)^2} \\ \frac{\partial f}{\partial y} &= \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2} \end{aligned}$$

Note that along the line $y = mx$, $\frac{\partial f}{\partial x} = \frac{2m^3}{(1+m^2)^2}$ and $\frac{\partial f}{\partial y} = \frac{1-m^2}{(1+m^2)^2}$ both of which does not go to zero (for $m \neq 1$). Hence the partial derivatives are not continuous. Thus the function is not differentiable at the origin.

Q3. *(15 marks)* Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a differentiable function. Let $a, b \in U$ such that U contains the line segment L from a to b (that is, $L = \{(1-t)a + tb : t \in [0, 1]\} \subset U$). Prove that there is some $c \in L$ such that $f(b) - f(a) = f'(c)(b - a)$.

Solution: This is Mean value theorem on higher dimensional spaces.
Consider the function $F : [0, 1] \to \mathbb{R}$ defined as $F(t) = f(a + t(b - a))$. Since f is differentiable, we have $f(x+h) = f(x) + D_x f(h) + |h|E_x(h)$, where $D_x f$ denotes the differential linear map of f at x and $E_x(h)$ goes to zero as h goes to zero. For each $h = (h_1, \cdots, h_n)$, since $D_x f$ is a linear map, as h going to zero, we have h_i goes to zero for all i. Thus $T_x(h) = \sum_{i=1}^n h_i T_x(e_i)$ goes to zero which imples f is continuous and hence F is continuous and differentiable.
Also by the chain rule, $F'(t) = f'(a + t(b - a))(b - a)$. By the mean value theorem on single variable, there exists $h \in [0, 1]$ such that $F(1) - F(0) = F'(h)$, that is, $f(b) - f(a) = f'(a + h(b - a))(b - a)$. Since $h \in [0, 1]$, we have $a + h(b - a) \in L$. Hence the proof.

Q4. *(10 marks)* Let $a, h \in \mathbb{R}^n, a + h \in B_r(a)$ and let $f : B_r(a) \to \mathbb{R}$ be a C^2 function. Define the real valued function η on $[0, 1]$ by $\eta(t) = f(a + th), t \in [0, 1]$. Compute the second order derivative of η.

Solution: We use Chain rule:

$$\begin{aligned} \eta'(t) &= Df(a + th).h \\ &= (\partial_1 f(a + th) \cdots \partial_n f(a + th)).(h_1 \cdots h_n)^t \\ &= \sum_{i=1}^n \partial_i f(a + th) h_i, \end{aligned}$$

where $\partial_i f(x_0) = \frac{\partial f(x)}{\partial x_i}\Big|_{x=x_0}$. Hence

$$\begin{aligned}\eta'(t+\epsilon) - \eta'(t) &= \sum_{i=1}^{n} \Big(\partial_i f(a+(t+\epsilon)h) - \partial_i f(a+th)\Big) h_i \\ \implies \eta''(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{ij} f(a+th) h_i h_j,\end{aligned}$$

where $\partial_{ij} f(x_0) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}\Big|_{x=x_0}$

Q5. *(10 marks)* Let X be a metric space. Prove that the closure of a connected subspace of X is also connected.

Solution: Suppose E is a connected subspace of X and $\overline{E}$ is not a connected subspace of X. There exists two sets A and B such that $\overline{E} = A \cup B$ and $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Without loss of generality, let A be a non-empty set. Suppose $a \in A \subset \overline{E}$. Since $A \cap \overline{B} = \emptyset$, there exists a neighborhood U such that $\overline{B} \cap U = \emptyset$. Also since $a \in \overline{E}$ there exists $x \in E \cap U$ and $x \notin B$. Hence $x \in E \cap A \neq \emptyset$. Since A and B are separated, $E \cap A$ and $E \cap B$ are separated. Thus we have $E = A \cup B$. Since A is non-empty and E is connected, B is an empty set. Hence the connectedness of $\overline{E}$.

Q6. *(10 marks)* Prove that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Solution: Suppose E is a compact subset of $\mathbb{R}$.
If E is unbounded, then there is an infinite bounded open cover. By compactness of E, there is a finite bounded open cover. Union of bounded sets is bounded and hence E must be bounded. If E is not closed, that is, if x is a limit point of E and does not belong to E, then $\mathbb{R}\backslash(x-\epsilon, x+\epsilon)$ for $\epsilon > 0$ forms an open cover of E. By compactness of E, there is a finite cover $\{\mathbb{R}\backslash(x=\epsilon_i, x+\epsilon_i)\}_{i=1,\cdots,N}$ for E. Let $\delta = min_i\{\epsilon_i\}$. $(x-\delta, x+\delta) \cap E = \emptyset$ implies that x is not a limit point. Hence E is closed.

Conversely, suppose E is closed and bounded. Hence there exists an interval $[a,b]$ such that $E \subset [a,b]$ and the complement E^c is open. Let $\mathcal{O}$ be an open cover for E. Then $\mathcal{O} \cup E^c$ is also a open cover of $[a,b]$. It is enough to prove that $[a,b]$ has a finite subcover.
Consider $S = \{x \in [a,b] : \mathcal{O} \cup E^c \text{ has a finite subcover for } [a,x]\}$. Since $\mathcal{O} \cup E^c$ is a cover for $\{a\}$, $a \in S$ and S is non-empty. Let $s = \sup S$. Since $[a,b]$ is closed, $s \in [a,b]$. Let $U \in \mathcal{O} \cup E^c$ be an open set such that $(s-\epsilon, s+\epsilon) \subset U$. By the definition of s, let $x \in S$ such that $x > s - \epsilon$. Hence there exists a finite subcover $\widetilde{\mathcal{O}} \subset \mathcal{O} \cup E^c$ of $[a,x]$. Let $\tilde{x} = \min\{x+\epsilon, b\}$. But by definition of s, $b < s+\epsilon$. Hence $\widetilde{\mathcal{O}} \cup U$ is finite subcover of $[a,b]$.

Q7. *(10 marks)* Let X be a compact metric space and $f : X \to X$ an isometry. Prove that $f(X) = X$.

Solution: f is an isometry, that is, $d(f(x), f(y)) = d(x, y)$. Hence f is continuous.Suppose $f(X) \subset X$ and $f(X) \neq X$. Let $x_0 \in X$ but $x_0 \notin f(X)$. Let $0 < \epsilon < d(x_0, f(X))$. Since $\{B_\epsilon(x)\}_{x \in X}$ is an open cover of X, by compactness there is a finite subcover $\{B_\epsilon(x_i)\}_{i=1,\cdots,N, x_i \in X}$ for X and N be the least number for such a cover to exist. Let $j \in \{1, \cdots, N$ such that $x_0 \in B_\epsilon(x_j)$. By the choice of ϵ, $B_\epsilon(x_j)$ does not intersect $f(X)$. By the continuity of f there are $N-1$ $\epsilon-$covers of $f(X)$ and hence for X which contradicts the minimality of N. Hence $f(X) = X$.

Q8. *(7+8 marks)*

(a) Let $t \in [0,1]$. Prove that the evaluation map $ev_t : C[0,1] \to \mathbb{R}$ defined by $ev_t(f) = f(t), f \in C[0,1]$ is continous.

Solution: Recall that the norm in $C[0,1]$ we consider is $\|f\|_\infty = sup_t|f(t)|$. Let $\epsilon > 0$ and $t \in [0,1]$ be given. Let $\{f_n\}$ be a sequence in $C[0,1]$ converging to $f \in C[0,1]$, that is, for every $\epsilon > 0$, there exists a large $N \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon$ for all $n \geq N$.

$$|ev_t(f_n) - ev_t(f)| = |f_n(t) - f(t)| \leq \sup_t |f_n(t) - f(t)| = \|f_n - f\|_\infty < \epsilon \ \forall n \geq N.$$

Hence the continuity of ev_t.

(b) Let K be a closed subset of $\mathbb{R}$ and $S_K := \{f \in C[0,1] : f([0,1]) \subset K\}$. Use part (a) to prove that S_K is a closed subset of $C[0,1]$.

Solution: Let f_n be a sequence in S_K and converge to f. Let $t \in [0,1]$ be arbitrary. Then $ev_t(f_n) \in K$ for all n. Since f_n converges to f; by (a), ev_t is continuous, $ev_t(f_n)$ is a sequence converging to $ev_t(f)$. Also, since K is a closed subset of $\mathbb{R}$, $ev_t(f) \in K$. Since this is true for all t, we have $f \in S_K$.

Q9. *(8+7 marks)* Let X be a metric space. Prove the following:

(a) The union of two intersecting connected subspaces of X is connected.

Solution: Suppose E and F are two intersecting connected subspaces of a metric space X and suppose $E \cup F$ is not connected. Then there exists two non-empty disjoint sets A, B such that $E \cup F = A \cup B$, $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. Let $x \in E \cap F$. Without loss of generality, let $x \in A$. Let $y \in B$, since B is non-empty. Without loss of generality, let $y \in E$. Then $x, y \in E$ implies $E \cap \tilde{A} = E \cap A \neq \emptyset \neq E \cap B = E \cap \tilde{B}$ where $\tilde{A} = E \cap A$ and $\tilde{B} = E \cap B$. But by connectedness of E, $E = \tilde{A} \cup \tilde{B}$ implies either $E \cap \tilde{A} = \emptyset$ or $E \cap \tilde{B} = \emptyset$. Thus we contradict to our hypothesis.

(b) The union of two compact subspaces of X is compact.

Solution: Let E, F be two compact subsets of the metric space X. Suppose $\{U_\alpha\}_\alpha$ is an open cover of $E \cup F$ and hence an open cover of both E and F. By compactness of E and F, let $\{U_i\}_{i=1,\cdots,N_1}$ and $\{V_i\}_{i=1,\cdots,N_2}$ be the respective finite subcovers of $\{U_\alpha\}$ for E and F. Then consider $\{U_i, V_j\}_{i=1,\cdots N_1, j=1,\cdots N_2}$. It is a finite subcover of $E \cup F$. Hence the proof.